On the Intensity of Light Reflected from Transparent Substances.

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A careful comparison of theory with experiment as regards the intensity of reflection would seem to suggest itself naturally as a crucial test of the validity of any optical theory. In spite of this, it was not till late in the last century that the problem was seriously undertaken by experimentalists. In 1870 Rood* turned his attention to the subject with the view of testing Fresnel's laws, and concluded from his experiments "that the reflecting power of glass conforms, in the closest manner, to the predictions of theory." However, in 1886, this conclusion was shown to be untenable by Lord Rayleigh.† The difficulties of measuring the intensity of the reflected light accurately are very considerable, and Rood had contented himself with estimating the transmitted light and deducing the amount that was reflected. Rayleigh showed that when this fact was considered the difference between Fresnel's formula and Rood's experimental results might amount to 7 per cent. of the reflected light, a difference much too great to be regarded as insignificant. Rayleigh found from his own experiments that recently polished glass has a reflecting power differing not more than 1 or 2 per cent. from Fresnel's formula; but that after some months or years the reflection may fall off 10 per cent. or more, and that without any apparent tarnish. About the same time Sir John Conroy carried out a lengthy series of experiments on the same subject. His results were published in the 'Phil. Trans.,' 1888, and confirmed those of Lord Rayleigh.

There can thus be no doubt of a decided departure from Fresnel's formula under certain circumstances. The difference is too great to be put down to experimental errors, and there is no evidence of such errors, seeing that the results of experiment are fairly consistent. Nor can there be very much doubt as to the direction in which to look for an explanation of the apparent divergence between theory and experiment. Everything points to a changing condition of the reflecting surface, and this suggests that a consideration of the layer of transition will show how Fresnel's laws are departed from in this as in some other directions. The object of the present paper is to investigate

* 'Amer. Journ. Science,' vols. 49 and 50.

† 'Roy. Soc. Proc.,' vol. 41; 'Scientific Papers,' vol. 2, p. 522.

this matter rather more systematically than appears to have been done hitherto.

In the first place it will be convenient to estimate the influence on the reflective power at normal incidence of a *uniform* layer on the surface of the reflecting substance. Let p be the frequency, c the velocity of light, and λ the wave-length in air, μ the refractive index of the reflecting substance, μ' that of the layer, and d its thickness. In the layer the components of the light vector are proportional to $e^{ip(t-\mu'x/c)}$. Hence a wave of unit amplitude will become of amplitude q after traversing the layer, where $q = e^{-ip\mu'd/c}$.

Thus,

$$q^2 = e^{-i\theta}, \quad \text{where} \quad \theta = 2p\mu'd/c = 4\pi\mu'd/\lambda = 2\mu'd_1, \quad \text{where} \quad d_1 = 2\pi d/\lambda.$$

If r_1 and r_2 be the factors of reflection on passing from air to the layer (μ') and from the layer to the reflecting substance (μ) respectively, then the reflected beam is represented by $\dfrac{r_1+q^2r_2}{1+q^2r_1r_2}$, where $q^2 = e^{-i\theta}$.

Hence if I be the intensity of the reflected light, we have

$$\mathrm{I} = \frac{r_1^2+r_2^2+2r_1r_2\cos\theta}{1+r_1^2r_2^2+2r_1r_2\cos\theta}, \quad \text{so that} \quad \cos\theta = \frac{\mathrm{I}(1+r_1^2r_2^2)-(r_1^2+r_2^2)}{2r_1r_2(1-\mathrm{I})}.$$

The values of r_1 and r_2 are given by Fresnel's formula, viz., $r_1 = \dfrac{\mu'-1}{\mu'+1}$, $r_2 = \dfrac{\mu-\mu'}{\mu+\mu'}$. If the thickness of the layer be so small that the *square* of d_1 can be neglected, then $\cos\theta = 1$ and $\mathrm{I} = \left(\dfrac{r_1+r_2}{1+r_1r_2}\right)^2$.

From the values of r_1 and r_2 given above we easily obtain

$$\frac{r_1+r_2}{1+r_1r_2} = \frac{\mu-1}{\mu+1}.$$

In this case, then, $\mathrm{I} = \left(\dfrac{\mu-1}{\mu+1}\right)^2$, which is the value of the intensity if there is no layer. It thus appears, as is to be expected for general reasons, that an exceedingly thin layer of transparent matter produces no effect on the intensity of the reflected light, and that in any case where a change is observed the thickness of the layer must be such that the square of $2\pi d/\lambda$ is appreciable.

If κ be the intensity when there is no layer, and if the layer diminishes the intensity by $1/n$, then we have

$$\mathrm{I} = \kappa\left(1-\frac{1}{n}\right) \quad \text{and} \quad \frac{r_1+r_2}{1+r_1r_2} = \sqrt{\kappa}.$$

Hence

$$\cos\theta = \frac{\kappa(1-n^{-1})(1+r_1^2 r_2^2)-(r_1+r_2)^2+2r_1r_2}{2r_1r_2(1-\kappa+\kappa n^{-1})}$$

$$= \frac{1-\kappa-\frac{1}{2}\kappa n^{-1}(r_1r_2+r_1^{-1}r_2^{-1})}{1-\kappa+\kappa n^{-1}}.$$

To obtain the minimum value of θ we have to make $r_1r_2+(r_1r_2)^{-1}$ a minimum. Now r_1r_2 vanishes when $\mu'=1$ and when $\mu'=\mu$, and r_1 and r_2 are both fractions, so that r_1r_2 is always less than unity. Thus we cannot have $r_1r_2=1$, which would make $r_1r_2+(r_1r_2)^{-1}$ a minimum. But we also get a minimum value of θ by making r_1r_2 a maximum. This makes $\mu'^2=\mu$ and $r_1=r_2=(\sqrt{\mu}-1)/(\sqrt{\mu}+1)$. This enables us to estimate approximately what must be the refractive index of the layer in order to produce a given diminution of intensity with the thinnest possible layer. We have $2\pi d/\lambda = d_1 = \theta/2\mu'$, and as θ is a minimum when $\mu'=\sqrt{\mu}$, we see that d will be a minimum for a value of μ' rather greater than $\sqrt{\mu}$. This result is independent of the thickness of the layer, but if this is not large a simple formula for the value of μ' that makes d a minimum is easily found.

We have

$$\cos 2\mu' d_1 = \cos\theta = \frac{1-\kappa-\frac{1}{2}\kappa n^{-1}(r_1r_2+r_1^{-1}r_2^{-1})}{1-\kappa+\kappa n^{-1}}.$$

Hence if d_1 be a minimum, we must also have

$$d_1\sin 2\mu' d_1 = \frac{\kappa(\mu+1)}{2n(1-\kappa)+2\kappa}\,\frac{\mu-\mu'^2}{(\mu'^2-1)(\mu^2-\mu'^2)}(r_1r_2-r_1^{-1}r_2^{-1}).$$

These equations may be written

$$\left(1-\kappa+\frac{\kappa}{n}\right)\cos\theta = 1-\kappa-\frac{\kappa}{n}\,\frac{\mu'^4+\mu'^2(\mu^2+4\mu+1)+\mu^2}{(\mu'^2-1)(\mu^2-\mu'^2)}, \tag{i}$$

and $$\left(1-\kappa+\frac{\kappa}{n}\right)\theta\sin\theta = -\frac{4\kappa}{n}\cdot\frac{(\mu+1)^2(\mu^2-\mu'^4)\,\mu'^2}{(\mu'^2-1)^2(\mu^2-\mu'^2)^2}. \tag{ii}$$

Now if we neglect *fourth* and higher powers of θ, we have $\cos\theta = 1-\frac{1}{2}\theta\sin\theta$, and substituting in (i) and (ii), we get

$$1-\kappa-\frac{\kappa}{n}\left[\frac{\mu'^4+\mu'^2(\mu^2+4\mu+1)+\mu^2}{(\mu'^2-1)(\mu^2-\mu'^2)}\right] = 1-\kappa+\frac{\kappa}{n}+\frac{2\kappa}{n}\cdot\frac{(\mu+1)^2\mu'^2(\mu^2-\mu'^4)}{(\mu'^2-1)^2(\mu^2-\mu'^2)^2},$$

which reduces to $$\mu'^4[\mu^2+1-2\mu'^2]=0,$$

so that the layer is thinnest when $\mu'^2 = \frac{1}{2}(\mu^2+1)$.

To illustrate the relation between the refractive index and the thickness of the layer required to effect a given diminution of the intensity, we have the following table, in which $\mu = 1{\cdot}5$ and the intensity is diminished 10 per cent. by the layer.

μ'.	1·1	1·2	1·225	1·25	1·275	1·3	1·4
d_1	0·3563	0·2749	0·2676	0·2637	0·2624	0·2643	0·3153

The thickness is least when $\mu = \sqrt{\frac{1}{2}(1+\mu^2)} = 1{\cdot}275$. For sodium light this would make the thickness of the layer rather less than 1/10000 of an inch.

This investigation of the influence of a uniform layer on the intensity shows that the value of d_1 required to produce a diminution of intensity such as was observed by Rayleigh and Conroy is not large. The hypothesis of a uniform layer in the cases dealt with by these experimenters is highly improbable, it being much more reasonable to suppose that there is a layer of gradual transition from one medium to the other. To investigate completely the effect of such a layer in the most general case would be a very troublesome matter, even if the law of variation of the refractive index were known—which is, of course, not the case. The problem, however, becomes comparatively simple* when the thickness of the layer is such that we can proceed by successive approximations, retaining different powers of d_1. The investigation of the simple case of a uniform layer suggests that this method is legitimate. The value of d_1 for a layer of gradual transition might be expected to be rather larger than for a uniform layer; but as d_1 for the uniform layer is little greater than 1/4, we should expect it to be considerably less than unity for the transition layer. The sequel will prove that this expectation is well founded and will justify the method of procedure.

We have already seen that for a layer of transparent matter it will be necessary to retain terms of the second order in d_1. The formulæ for such a case are given on pp. 59 and 60 of the paper just referred to.

If ϕ be the angle of incidence, μ the index of refraction, and $\sin\phi = \mu\sin\phi'$ then—except very near the polarising angle—the amplitude of the reflected wave for vibrations parallel to the plane of incidence is given by the formula

$$R = \frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'}\left[1 + \frac{2\cos\phi\cos\phi'}{(\mu^2\cos^2\phi - \cos^2\phi')^2}(a + b\sin^2\phi + c\sin^4\phi)\right].$$

Similarly, for vibrations perpendicular to the plane of incidence we have

$$R' = \frac{\mu\cos\phi' - \cos\phi}{\mu\cos\phi' + \cos\phi}\left[1 + \frac{2a\cos\phi\cos\phi'}{(\mu^2\cos^2\phi' - \cos^2\phi)^2}\right],$$

* See 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 49.

where

$$a = \mu^3 d_1^2 [E^2\mu^2+1-2E-2H(\mu^2-1)],$$

$$b = \mu^3 d_1^2 [-E^2(1+\mu^2)-2(F-E)\mu^{-2}+EF(1+\mu^2)\mu^{-2}-(L-J)(1-\mu^{-2}) + 2H(\mu^2-\mu^{-2})],$$

$$c = \mu^3 d_1^2 [E^2+F^2\mu^{-4}-EF(1+\mu^{-4})+(L-J)(1-\mu^{-4})].$$

The addition to the intensity due to the layer is

$$2\cos\phi\cos\phi'\left[\frac{a+b\sin^2\phi+c\sin^4\phi}{(\mu\cos\phi+\cos\phi')^4}+\frac{a}{(\mu\cos\phi'+\cos\phi)^4}\right];$$

so that, as far as the correction to the intensity is concerned, everything can be expressed in terms of *three* constants a, b, and c, depending on the thickness of the layer and the law of distribution of μ' within the layer. At normal incidence the correction is $2a/(\mu+1)^4$, and this will be positive or negative according to the sign of a, that is, according as H is less or greater than $(E^2\mu^2+1-2E)/2(\mu^2-1)$.

In order to test these results, we shall compare them with Conroy's experiments* on the amount of light reflected by Crown glass (before repolishing) at various incidences.

On plotting Conroy's results, it is seen that they are very fairly consistent, except that the mean of his measurements at 40° is somewhat high, and that at 70° rather low. This is probably due to the fact that his estimate of the intensity of reflection at these two angles was made under different circumstances than at most of the other incidences. In most cases he made experiments with his photometer in two positions (A and B), and at 50°, 60°, and 65° the B readings are uniformly higher than the A ones. If we assume that the same would have been the case at 40° and 70°, we shall have to depress his result at 40° in a certain ratio, and raise that at 70°, for at the former angle the readings were all in the position B, and at the latter in the position A.

The value of μ was found by Conroy to be 1·5145. For the other constants a, b, and c appearing in the expression for the intensity, and due to the presence of the layer, we shall take tentatively—

$$a = -0{\cdot}041, \quad b = 0{\cdot}06642, \quad c = -0{\cdot}08079.$$

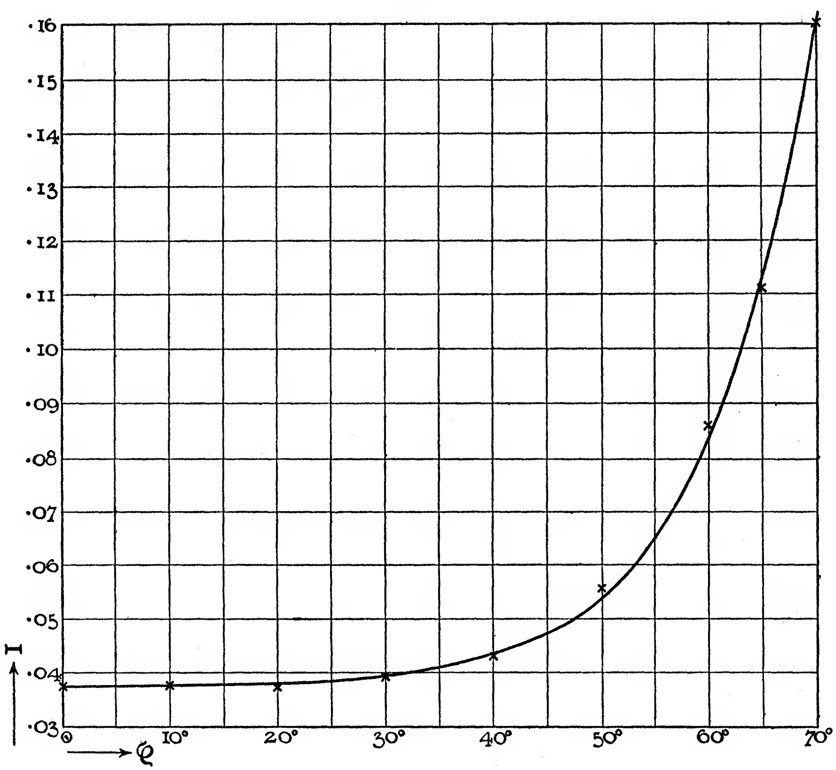
With these we derive the following table, giving the intensity of reflection at different angles of incidence, and a comparison with the results of Conroy's experiments:—

* 'Phil. Trans.,' A, vol. 180, p. 271.

ϕ =	0.	10°.	20°.	30°.	40°.	50°.	60°.	65°.	70°.
Theory	0·0378	0·0378	0·0380	0·0392	0·0439	0·0537	0·083[illegible]	0·1128	0·1612
Experiment ...	0·0378	0·0378	0·0377	0·0392	0·0437	0·0553	0·0854	0·1116	0·1604
Difference ...	0	0	+0·0003	0	+0·0002	−0·0016	−0·0023	+0·0012	+0·0008

In this table the experimental results at 40° and 70° have been "corrected" in accordance with the principle explained above. The results are exhibited graphically in fig. 1 below, and it will be seen from this figure, or from the table above, that the agreement between theory and experiment is as close as could be desired. The continuous curve corresponds to the theory, the crosses represent the experimental results.

Fig. 1.



We have seen that if we are concerned only with the intensity of the reflected light, everything can be determined satisfactorily in terms of three constants *a*, *b*, and *c*. If, however, we wish to discuss all the circumstances of the case, *i.e.*, not merely the intensity of the reflected light, but also the position of the polarising angle, the change of phase produced by reflection,

and the amplitude and phase of the transmitted beam, then there are six constants concerned—d_1, E, F, H, J, and L. All of these, except d_1, could be calculated by integration for a given law of refractive index (μ') within the layer.

The converse problem of determining the law of μ' to fit in with the experimental results of any given case is, of course, indeterminate. Theoretically we might obtain a number of terms in the expression of μ'^2 by taking, in the most general case,

$$\mu'^2 = (1+p_1x)(1+p_2x)(1+p_3x)(1+p_4x)(1+p_5x)(1+p_6x).$$

The integrations involved in the calculation of E, F, H, J, and L could all be carried out, and a comparison of theory with the experimental measure of six different quantities would give sufficient equations to determine the unknown constants p. It is obvious, however, that the algebraical equations thus obtained would be very complicated, if not quite intractable. If we wish to form some estimate of the law of variation of the refractive index within the layer, it will probably be much simpler to calculate the constants for various simple laws in the hope of reaching one that fits in well with the facts.

As an example of this method, and to see in what way the various constants depend on the law of μ' within the layer, we shall consider the case presented by Conroy's experiments now under discussion. If the layer is of thickness d_2 we shall regard it as extending from $x=0$ to $x=1$, so that xd is the distance of a point from one boundary of the layer.* The various constants will be determined for different laws of μ', *e.g.*, for $\mu'^2=1+px$, the constant p being chosen in each case so as to make $\mu'=\mu=1·5145$ at the boundary of the layer where $x=1$. In order to give results agreeing with experiment when the incidence is normal, we must take the constant $a=0·041$. This will determine the thickness of the layer (d) and the quantity d_1 ($=2\pi d/\lambda$), in each case, in virtue of the formula for a on p. 22 above. The results are set out in the following table :—

	μ'^2.	p.	E.	H.	F.	J.	L.	$+b$.	$-c$.	d_1.
I	$1+px$	μ^2-1	0·7178	0·3120	1·472	0·3886	0·6679	0·06512	0·06958	0·4386
II	$1/(1+px)$	$1/\mu^2-1$	0·6415	0·2770	1·647	0·3886	0·6679	0·06971	0·07665	0·4606
III	$1+px^2$	μ^2-1	0·6240	0·2650	1·714	0·3805	0·6885	0·03859	0·07718	0·5406
IV	$1/(1+px^2)$	$1/\mu^2-1$	0·5661	0·2455	1·863	0·3975	0·6573	0·1201	0·08175	0·6039
V	$(1+px)^{\frac{1}{2}}$	μ^4-1	0·7546	0·3320	1·393	0·3950	0·6570	0·06527	0·06579	0·4341
VI	$(1+px)^{-\frac{1}{2}}$	$1/\mu^4-1$	0·6071	0·2637	1·732	0·3950	0·6570	0·06642	0·08079	0·4810
VII	$(1+px^2)^{\frac{1}{2}}$	μ^4-1	0·6554	0·2780	1·636	0·3770	0·6949	0·03281	0·07381	0·5135
VIII	$(1+px^2)^{-\frac{1}{2}}$	$1/\mu^4-1$	0·5419	0·2382	1·926	0·4070	0·6362	0·03599	0·08921	0·6655
IX	$(1+px)^2$	$\mu-1$	0·6987	0·3024	1·5145	0·3868	0·6714	0·1341	0·08569	0·4863

* Compare 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 53. What is there denoted by x_1 is here called x.

With the values of the constants a, b, and c derived from this table we can calculate the intensity of the light reflected at different incidences for the various laws considered. The following table exhibits the results of this calculation :—

ϕ =	0.	10°.	20°.	30°.	40°.	50°.	60°.	65°.	70°.
I	0·0378	0·0378	0·0380	0·0392	0·0430	0·0540	0·0838	0·1138	0·1626
II	0·0378	0·0378	0·0380	0·0392	0·0431	0·0541	0·0837	0·1137	0·1621
III	0·0378	0·0378	0·0378	0·0387	0·0421	0·0523	0·0806	0·1096	0·1571
IV	0·0378	0·0379	0·0383	0·0400	0·0446	0·0568	0·0884	0·1196	0·1699
V	0·0378	0·0378	0·0380	0·0392	0·0431	0·0541	0·0841	0·1142	0·1632
VI	0·0378	0·0378	0·0380	0·0392	0·0439	0·0537	0·0831	0·1128	0·1612
VII	0·0378	0·0378	0·0378	0·0388	0·0429	0·0520	0·0803	0·1092	0·1567
VIII	0·0378	0·0378	0·0378	0·0387	0·0428	0·0517	0·0795	0·1080	0·1550
IX	0·0378	0·0379	0·0384	0·0402	0·0450	0·0575	0·0875	0·1210	0·1725

The differences between these results and those of Conroy's experiments are as follows :—

ϕ =	0.	10°.	20°.	30°.	40°.	50°.	60°.	65°.	70°.
I	0	0	+0·0003	0	−0·0007	−0·0013	−0·0016	+0·0022	+0·0022
II	0	0	+0·0003	0	−0·0006	−0·0012	−0·0017	+0·0021	+0·0017
III	0	0	+0·0001	−0·0005	−0·0016	−0·0030	−0·0048	−0·0020	−0·0033
IV	0	+0·0001	+0·0006	+0·0008	+0·0009	+0·0015	+0·0030	+0·0080	+0·0095
V	0	0	+0·0003	0	−0·0006	−0·0011	−0·0013	+0·0026	+0·0028
VI	0	0	+0·0003	0	+0·0002	−0·0016	−0·0023	+0·0012	+0·0008
VII	0	0	+0·0001	−0·0004	−0·0008	−0·0033	−0·0051	−0·0024	−0·0037
VIII	0	0	+0·0001	−0·0005	−0·0009	−0·0036	−0·0059	−0·0036	−0·0054
IX	0	+0·0001	+0·0007	+0·0010	+0·0013	+0·0022	+0·0041	+0·0094	+0·0121

An examination of this table shows that Law VI, according to which $\mu'^2 = (1+px)^{-\frac{1}{2}}$, agrees best with the experimental results, and this is the case represented in fig. 1 above. It will be seen, however, that some of the other laws, *e.g.*, I and II, represent the intensity almost equally well. There are, however, other quantities than the intensity of the reflected light that have to be considered. One of these, recorded by Conroy, is the magnitude of the polarising angle. This will depend on the law of variation of μ' within the layer, and we shall proceed to obtain a formula setting out the nature of this dependence.

For vibrations parallel to the plane of incidence we have the following formula for r, the vector representing the reflected beam :—*

$$r = \frac{(\mu\cos\phi - \cos\phi') + i\mu d_1(\mathrm{E}\mu\cos\phi\cos\phi' - 1 + \mathrm{F}\mu^{-2}\sin^2\phi)}{(\mu\cos\phi + \cos\phi') + i\mu d_1(\mathrm{E}\mu\cos\phi\cos\phi' + 1 - \mathrm{F}\mu^{-2}\sin^2\phi)}.$$

* See 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 55.

At Brewster's angle we have $\mu \cos\phi - \cos\phi' = 0$, so that in its neighbourhood the intensity of the reflected light is

$$I = |r|^2 = \left(\frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'}\right)^2 + \mu^2 d_1^2 \left(\frac{E\mu\cos\phi\cos\phi' - 1 + F\mu^{-2}\sin^2\phi}{\mu\cos\phi + \cos\phi'}\right)^2$$

$$= R^2 + \rho^2,$$

where

$$R = \frac{\mu\cos\phi - \cos\phi'}{\mu\cos\phi + \cos\phi'} \quad \text{and} \quad \rho = \mu d_1 \frac{E\mu\cos\phi\cos\phi' - 1 + F\mu^{-2}\sin^2\phi}{\mu\cos\phi + \cos\phi'}.$$

The intensity is a minimum when $R\dfrac{dR}{d\phi} + \rho\dfrac{d\rho}{d\phi} = 0$; now, $\rho\dfrac{d\rho}{d\phi}$ is of the *second* order of small quantities, so that if we neglect terms of this order we have $R\,(dR/d\phi) = 0$, which is satisfied by $R = 0$, *i.e.*, $\mu\cos\phi - \cos\phi' = 0$. This gives $\phi = \alpha = \tan^{-1}\mu$, *i.e.*, Brewster's angle, so that, to the first order of small quantities, the layer has no influence on the position of the polarising angle. If, however, we include terms of the second order in d_1, then I will be a minimum when $\phi = \alpha - x$, where x is a quantity of the second order in d_1. Using the approximation $f(\phi) = f(\alpha - x) = f(\alpha) - xf'(\alpha)$, we get the following equation to determine x—

$$\left(R - x\frac{dR}{d\phi}\right)\left(\frac{dR}{d\phi} - x\frac{d^2R}{d\phi^2}\right) + \left(\rho - x\frac{d\rho}{d\phi}\right)\left(\frac{d\rho}{d\phi} - x\frac{d^2\rho}{d\phi^2}\right) = 0,$$

in which ϕ is to be put equal to α after differentiating.

Since $R = 0$ when $\phi = \alpha$, this gives

$$x = \rho\frac{d\rho}{d\phi}\bigg/\left(\frac{dR}{d\phi}\right)^2 \text{ radians,}$$

$$\textit{i.e.,}\quad x = \frac{\mu^3 d_1^2}{2(\mu^2+1)(\mu^4-1)^2}[1+\mu^2 - F - E\mu^2]$$
$$\times[(1+\mu^2)(1+\mu^4) - F(1+4\mu^2+\mu^4) + E\mu^2(1+\mu^4)].$$

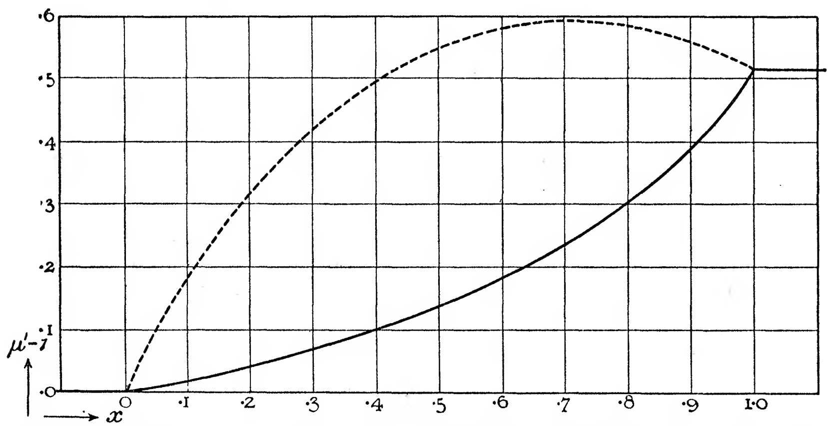
This is the *diminution* of the polarising angle, and the formula shows that the layer will not always diminish the polarising angle along with the intensity of reflection, but may increase the one and diminish the other.

The values of x (expressed in minutes and seconds) for the nine laws considered above, and the corresponding polarising angle $(\alpha - x)$, are as follows:—

Law ...	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	IX.
	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″	° ′ ″
x	0 27 38	0 16 24	0 13 30	0 0 0·58	0 31 40	0 10 12	0 19 33	−7 16	0 30 15
$\alpha - x$	56 5 12	56 17 26	56 20 20	56 33 49	56 2 10	56 23 38	56 14 18	56 41 6	56 3 35

The mean of Conroy's estimates of the polarising angle with this particular glass was 56° 23′ 30″. This agrees most closely with the deduction from Law VI, the difference being only 8 seconds. This law was found to fit in best with the experimental measures of the intensity, so that it is probably a close approximation to the truth. The corresponding values of μ' are represented graphically in the continuous curve of fig. 2 below. A similar representation of Law II would show that the values of μ' for these two laws are not very different, and a glance at the tables above makes it evident that Law II would represent the facts almost as well as Law VI.

FIG. 2.



Conroy's experiments on the amount of light transmitted by glass plates of different thicknesses, and under different conditions of polish, agree well with his estimates of the reflecting power. They prove that the departure from Fresnel's law is certainly not due to an irregular scattering of the light at the surface, as has sometimes been suggested. That this is not the explanation of the apparent discrepancy between theory and experiment is also manifest from the fact that both Rayleigh and Conroy found that freshly polished surfaces reflect *more* than Fresnel's formula indicates. Conroy's experiments, immediately after repolishing, were not so numerous as those previously discussed. The results are consequently not nearly so consistent as before, and we cannot build much upon them as regards the law of variation of the refractive index. It will, perhaps, suffice to take a single law, as an illustration of a possible condition of affairs.

Let $\mu'^2 = 1+\alpha x-\beta x^2$, $\alpha = \beta+\mu^2-1$. Then

$$\mathrm{E} = \frac{1}{\mu^2}\int_0^1 \mu'^2\,dx = \frac{1}{\mu^2}\left[\frac{1}{2}+\frac{\beta}{6}+\frac{\mu^2}{2}\right]; \quad \mathrm{H} = \frac{1}{\mu^2}\int_0^1 dx\int_0^x \mu'^2 dx = \frac{1}{\mu^2}\left[\frac{1}{3}+\frac{\beta}{12}+\frac{\mu^2}{6}\right]$$

$$\frac{a}{\mu^3 d_1^2} = \mathrm{E}^2\mu^2+1-2\mathrm{E}-2\mathrm{H}\,(\mu^2-1) = \frac{1}{36\mu^2}\,[\beta^2-3-3\mu^2\,(\mu^2-2)];$$

so that

$$\beta^2 = 3+3\mu^2\,(\mu^2-2)+36ad_1^{-2}.$$

In order to increase the reflective power, we must have a positive, and this requires β to be greater than $\{3+3\mu^2(\mu^2-2)\}^{\frac{1}{2}}$, or, in the present case, $\beta > 2{\cdot}24$.

By way of trial we shall take $\beta = 3$, and we then get

$$E = 0{\cdot}9358; \quad H = 0{\cdot}4209; \quad F = 1{\cdot}132; \quad L-J = 0{\cdot}2210.$$

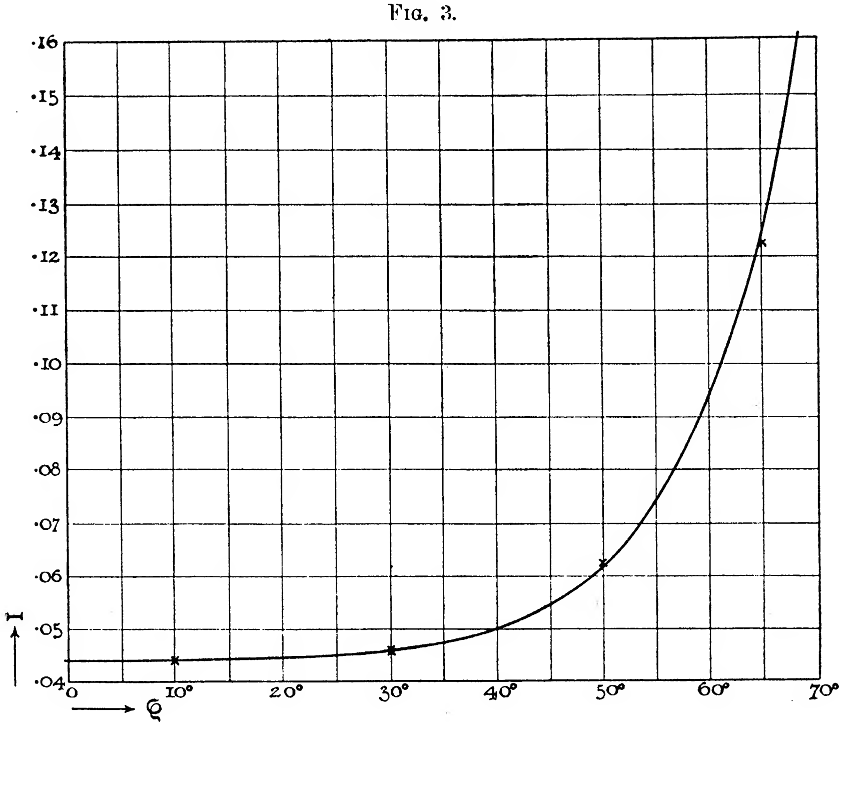
Taking the correction to the intensity at normal incidence to be 0·0022, the following are the values for the various constants :—$a = 0{\cdot}022$ $b = -0{\cdot}0434$; $c = 0{\cdot}0170$; $d_1 = 0{\cdot}3625$.

The *diminution* of the polarising angle, calculated from the formula of p. 26, is 1′ 5″, so that the polarising angle is 56° 32′ 45″, which agrees exactly with the mean of Conroy's measurements. The experimental estimates of the intensity were most consistent at incidences of 10°, 30°, 50°, 61°, and 75°. The following table sets out the results and compares them with those obtained from theory with the constants given above :—

ϕ =	10°.	30°.	50°.	65°.	75°.
Theory	0·0441	0·0455	0·0617	0·1248	0·2571
Experiment	0·0441	0·0455	0·0627	0·1228	0·2633
Difference	0	0	−0·0010	+0·0020	−0·0062

The results are also represented graphically in fig. 3 below.

Fig. 3.



Considering the uncertainty of the experimental results, the agreement is as close as could be expected. The dotted curve in fig. 2 above represents the march of μ' within the layer.

So far we have dealt solely with the influence of a perfectly transparent layer on the intensity of reflection and the position of the polarising angle. We have verified that such a layer, whether uniform or continuously varying, has no effect, if it is so thin that squares and higher powers of d_1 can be neglected. This need not be the case if the layer were of absorbing material, and it may be worth while to consider briefly what would be the effect of such a layer.

At normal incidence, when $\phi = 0$, we have, with our previous notation,

$$r = \frac{(\mu-1)+id_1\mu(\mathrm{E}\mu-1)}{(\mu+1)+id_1\mu(\mathrm{E}\mu+1)}$$

$$= \frac{\mu-1}{\mu+1}\left[1+\frac{i2\mu^2d_1}{\mu^2-1}(\mathrm{E}-1)\right] \text{(neglecting squares and higher powers of } d_1\text{)}$$

$$= \mathrm{R}(1+\alpha e^{i\beta}).$$

Thus the amplitude of the reflected wave is increased by multiplying by the factor $1+\alpha\cos\beta$.

In the layer, let $\mu' = \mathrm{M}e^{-i\gamma}$, then we have

$$\mathrm{E} = \frac{1}{\mu^2}\int_0^1 \mathrm{M}^2\cos 2\gamma\, dx - \frac{i}{\mu^2}\int_0^1 \mathrm{M}^2\sin 2\gamma\, dx = \mathrm{A}-i\mathrm{B}, \text{ say.}$$

And $\alpha\cos\beta$ is the real part of $i(\mathrm{E}-1)\,.\,2\mu^2d_1/(\mu^2-1)$, and is therefore equal to $2\mathrm{B}\mu^2d_1/(\mu^2-1)$.

If the layer were transparent throughout we should have $\gamma = 0$ and therefore $\mathrm{B} = 0$, so that, to this order, as we have seen, there would be no change of intensity. For an absorbing medium, however, γ will not be zero, but will be less than $\frac{1}{2}\pi$, so that $\sin 2\gamma$ will necessarily be positive and B consequently positive. Thus the effect of such a layer will be to increase the reflecting power. The observed departure from Fresnel's formula for a freshly polished surface might, then, be caused by a very thin layer of absorbing material, due either to the polishing powder or to contamination with some foreign substance of a greasy nature. Rayleigh* showed that the deviation of the ellipticity from Fresnel's formula in the neighbourhood of the polarising angle was due to greasy contamination. In a later paper† he followed out the same idea when dealing with the light reflected from water at nearly perpendicular incidence. The reflection actually observed, even after the surface was cleaned, was about $1\frac{1}{2}$ per cent.

* 'Phil. Mag.,' 1892; 'Scientific Papers,' vol. 3, p. 496.

† 'Phil. Mag.,' 1892; 'Scientific Papers,' vol. 4, p. 3.

greater than that given by Fresnel's formula. "The disagreement is too small a foundation upon which to build with any confidence," but from what we have seen a very slight residual contamination might help to bridge the slight difference between theory and observation.

If we are dealing with incidences other than normal, we have to distinguish between vibrations parallel and perpendicular to the plane of incidence. In both cases we have equations of the form $r = \mathrm{R}(1+\kappa)$, where κ is a small quantity of the first order in d_1. The increase of the intensity due to the layer is $\kappa_1 \mathrm{R}_1{}^2 + \kappa_2 \mathrm{R}_2{}^2$. Here κ_1 is the real part of

$$id_1 \left[\frac{\mathrm{E}\mu^2 \cos\phi \cos\phi' - \mu + \sin^2\phi \,.\, \mathrm{F}\mu^{-1}}{\mu\cos\phi - \cos\phi'} - \frac{\mathrm{E}\mu^2\cos\phi\cos\phi' + \mu - \sin^2\phi \,.\, \mathrm{F}\mu^{-1}}{\mu\cos\phi + \cos\phi'}\right]$$

$$= \frac{2id_1\cos\phi}{\mu^2\cos^2\phi - \cos^2\phi'}\left[\mu^2(\mathrm{E}-1) + (\mathrm{F}-\mathrm{E})\sin^2\phi\right].$$

Similarly, κ_2 is the real part of

$$\frac{2id_1\cos\phi \,.\, \mu^2(\mathrm{E}-1)}{\mu^2\cos^2\phi' - \cos^2\phi}.$$

The increase of the intensity is therefore

$$\kappa\cos\phi\left[\frac{\mu\cos\phi - \cos\phi'}{(\mu\cos\phi + \cos\phi')^3}(1 - e\sin^2\phi) + \frac{\mu\cos\phi' - \cos\phi}{(\mu\cos\phi' + \cos\phi)^3}\right],$$

where κ and e are constants depending on the thickness of the layer and the law of the distribution of μ' within it. As a numerical illustration we may choose κ and e, so as to give the same results as those found for the transparent layer immediately after repolish. The consequent intensities and their comparison with Conroy's observations are set out in the table below:—

ϕ =	10°.	30°.	50°.	65°.	75°.
Theory	0·0441	0·0457	0·0622	0·1248	0·2562
Experiment	0·0441	0·0455	0·0627	0·1228	0·2633
Difference	0	+0·0002	−0·0005	+0·0020	−0·0071

The differences between the results for the absorbing and for the transparent layer are not sufficiently marked to enable us to decide definitely between the two hypotheses merely on the ground of the reflective power. To settle such a matter conclusively we should require much more accurate experimental results than are now at our disposal.